The π^{-1} matrix is

$$
\pi^{-1} = \begin{array}{cc}
\begin{array}{cccccccccccc} w_1 & w_{x_1} & w_{y_1} & w_2 & w_{x_2} & w_{y_2} & w_3 & w_{x_3} & w_{y_3} & w_4 & w_{x_4} & w_{y_4} \end{array} & \\
\begin{bmatrix}
\frac{1}{4} & -\frac{a}{8} & \frac{b}{8} & \frac{1}{4} & -\frac{a}{8} & -\frac{b}{8} & \frac{1}{4} & \frac{a}{8} & -\frac{b}{8} & \frac{1}{4} & \frac{a}{8} & \frac{b}{8} \\
\frac{3}{8a} & -\frac{1}{8} & \frac{b}{8a} & \frac{3}{8a} & -\frac{1}{8} & -\frac{b}{8a} & -\frac{3}{8a} & -\frac{1}{8} & \frac{b}{8a} & -\frac{3}{8a} & -\frac{1}{8} & -\frac{b}{8a} \\
-\frac{3}{8b} & \frac{a}{8b} & -\frac{1}{8} & \frac{3}{8b} & -\frac{a}{8b} & -\frac{1}{8} & \frac{3}{8b} & \frac{a}{8b} & -\frac{1}{8} & \frac{3}{8b} & -\frac{a}{8b} & -\frac{1}{8} \\
0 & \frac{1}{4a} & 0 & 0 & \frac{1}{4a} & 0 & 0 & -\frac{1}{4a} & 0 & 0 & -\frac{1}{4a} & 0 \\
-\frac{1}{2ab} & \frac{1}{8b} & -\frac{1}{8a} & \frac{1}{2ab} & -\frac{1}{8b} & -\frac{1}{8a} & -\frac{1}{2ab} & -\frac{1}{8b} & \frac{1}{8a} & \frac{1}{2ab} & \frac{1}{8b} & \frac{1}{8a} \\
0 & 0 & -\frac{1}{4b} & 0 & 0 & \frac{1}{4b} & 0 & 0 & \frac{1}{4b} & 0 & 0 & -\frac{1}{4b} \\
-\frac{3}{4a^3} & \frac{3}{4a^2} & 0 & -\frac{3}{4a^3} & \frac{3}{4a^2} & 0 & \frac{3}{4a^3} & \frac{3}{4a^2} & 0 & \frac{3}{4a^3} & \frac{3}{4a^2} & 0 \\
0 & -\frac{1}{4ab} & 0 & 0 & \frac{1}{4ab} & 0 & 0 & -\frac{1}{4ab} & 0 & 0 & \frac{1}{4ab} & 0 \\
0 & 0 & -\frac{1}{4ab} & 0 & 0 & \frac{1}{4ab} & 0 & 0 & -\frac{1}{4ab} & 0 & 0 & \frac{1}{4ab} \\
\frac{3}{4b^3} & 0 & \frac{3}{4b^2} & -\frac{3}{4b^3} & 0 & \frac{3}{4b^2} & -\frac{3}{4b^3} & 0 & \frac{3}{4b^2} & \frac{3}{4b^3} & 0 & \frac{3}{4b^2} \\
\frac{3}{4a^3b} & -\frac{3}{4a^2b} & 0 & -\frac{3}{4a^3b} & \frac{3}{4a^2b} & 0 & \frac{3}{4a^3b} & \frac{3}{4a^2b} & 0 & -\frac{3}{4a^3b} & -\frac{3}{4a^2b} & 0 \\
\frac{3}{4ab^3} & 0 & \frac{3}{4ab^2} & -\frac{3}{4ab^3} & 0 & \frac{3}{4ab^2} & \frac{3}{4ab^3} & 0 & -\frac{3}{4ab^2} & -\frac{3}{4ab^3} & 0 & -\frac{3}{4ab^2}
\end{bmatrix} &
\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \end{array}
\end{array}
$$

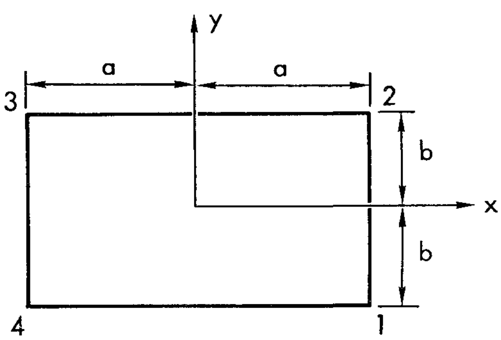

Fig. 1 Plate element.

Then, for example,

$$\varphi_1 = \tfrac{1}{4} + (3/8a)\,x - (3/8b)y - (1/2ab)xy -$$
$$(1/8a^3)x^3 + (1/8b^3)y^3 + (1/8a^3b)x^3y + (1/8ab^3)xy^-$$

With the origin of the plate element at the center, the functions that are even in either x or y, after integration, will vanish upon evaluation and so need not be retained in the integrand.

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Order of a Perturbation Method

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Introduction

It is shown that accuracies usually denoted as "second order" (or higher) can be achieved by repeated application of "first-order" expressions, rather than by derivation of second-order expressions. The difference in approach is analogous to that between existence proofs for solutions of differential equations using Picard iterants instead of dominating series. A numerical example is given. It is postulated that one reason the second-order expressions derived for "Telstar" orbit prediction[2] are apparently at least ten times more accurate than other second-order methods[5] is this difference in the method of application of the expressions.

Proof

Equations of the type

$$da_i/dt = \kappa G_i(a_j,t) \qquad i,j, = 1,\ldots,6 \tag{1}$$

where κ is a small parameter, arise quite naturally in celestial

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mechanics.† Usually it is desirable for analytic reasons to use an angular independent variable, although an angle-time relationship must always be available to maintain contact with the real world. Thus, we will rewrite (1) as

$$da_i/d\theta = \kappa F_i(a_j,\theta) \qquad i,j, = 1,\ldots,6 \tag{2}$$

We will assume that the angle-time (θ-t) relationship is exact and take (2) as the system of equations under consideration. [In practice, the errors introduced by the time relation can be maintained at a higher order than the errors introduced in the solution of (2).] We are also given the initial conditions

$$a_i(\theta_0) = c_i \qquad i = 1,\ldots,6 \tag{3}$$

A standard procedure, originating with Euler, is to express the solution of (2) as a power series in κ:

$$a_i(\theta) = c_i + \kappa a_i^{(1)}(\theta) + \kappa^2 a_i^{(2)}(\theta) + \ldots + \kappa^q a_i^{(q)}(\theta) + \ldots \tag{4}$$

where the functions $a_i^{(q)}$ satisfy differential equations

$$da_i^{(1)}/d\theta = F_i(c_j,\theta)$$
$$\frac{da_i^{(2)}}{d\theta} = \sum_{j=1}^{6} \frac{\partial F_i}{\partial a_j} a_j^{(1)}, \text{ etc.} \tag{5}$$

which may be solved sequentially, by quadratures.

It can be shown[1] that, under suitable conditions on the F_i and for certain ranges of κ and $\theta - \theta_0$, the series (4) converges to the solution of (2). Moulton (Ref. 1, Chap. III) gives the details of these proofs and also references to the original works of Cauchy and Poincaré. The representation of the solution in the form (4) is of interest, not only as the basis of an existence proof, but as a practical method of solution of systems of equations such as (2) which arise during an analysis of the orbit of an earth satellite.[2–4]

As mentioned previously, (5) can be solved sequentially by quadratures, i.e., we can obtain the $a_i^{(q)}(\theta)$ as

$$a_i^{(1)}(\theta) = \int_{\theta_0}^{\theta} F_i(c_j,\tilde{\theta})\, d\tilde{\theta} \tag{6}$$

$$a_i^{(q)}(\theta) = \int_{\theta_0}^{\theta} p_j^{(q)}(c_j,a_j^{(1)},\ldots,a_j^{(q-1)},\tilde{\theta})\, d\tilde{\theta} \tag{7}$$

If we approximate the solution to (2) by

$$a_i(\theta) = c_i + \kappa a_i^{(1)}(\theta) \tag{8}$$

where $a_i^{(1)}$ is given by (6), we say we have included the "first-order" perturbations. If we include an additional term

$$a_i(\theta) = c_i + \kappa a_i^{(1)}(\theta) + \kappa^2 a_i^{(2)}(\theta) \tag{9}$$

we say we have included the second-order perturbations. [Often the F_i are periodic in θ, and it is simplest to obtain the $a_i^{(q)}$ for values of $\theta = \theta_n = \theta_0 + 2n\pi$, where n is an integer. If (8) or (9) were written for $\theta = \theta_1$, for instance, we would say that they included the "first-order secular" or "second-order secular" perturbations, respectively.]

An alternate method of expressing the solution to (2) is as an integral equation

$$a_i(\theta) = c_i + \kappa \int_{\theta_0}^{\theta} F_i(a_j(\tilde{\theta}),\tilde{\theta})\, d\tilde{\theta} \tag{10}$$

This expression can be replaced by a sequence of the form

$$a_i^{(q)}(\theta) = c_i + \kappa \int_{\theta_0}^{\theta} F_i(a_i^{(q-1)}(\tilde{\theta}),\tilde{\theta})\, d\tilde{\theta} \tag{11}$$

where $a_i^{(0)} = c_i$. This forms the basis of the existence proof using Picard iterants. (See Moulton, Chap. XI, for details and a reference to the original work of Picard.) Of more interest to us here, is the fact that (10) is an exact solution to (2), which we can solve to any degree of approximation we desire by replacing the integral by a modified Riemann sum, i.e.,

$$a_i(\theta) = c_i + \kappa \sum_{k=0}^{N} \int_{\theta_k}^{\theta_{k+1}} F_i[a_j(\theta_k),\tilde{\theta}]\, d\tilde{\theta} \tag{12}$$

where $\theta_{N+1} = \theta$. Equation (12) would be exact except that we have replaced $a_j(\tilde{\theta})$ with $a_j(\theta_k)$ in F_i. As $N \to \infty$, (12) $\to$ (10). Also, we can make (12) as accurate as desired by increasing N. (These last two statements are true if the F_i are continuous; this must be true if either the dominating series or Picard iterant existence proofs are to be valid.[1])

Now (12) can be written as

$$a_i(\theta) = c_i + \kappa \sum_{k-0}^{N} a_i^{(1)}(\theta_{k+1}) \tag{13}$$

where

$$a_i^{(1)}(\theta_{k+1}) = \int_{\theta_k}^{\theta_{k+1}} F_i[a_j(\theta_k),\tilde{\theta}]\, d\tilde{\theta} \tag{14}$$

Equation (14) is exactly analogous to (6). Thus, by repeated updating of the lower limits of integration in the first-order perturbation expressions, we can obtain accuracies equivalent to second-order (or higher‡) perturbation methods.

Table 1 Perturbation in nodal angle and argument of perigee

	$-\Delta\Omega$ (10^{-3} rad)	$\Delta\omega$ (10^{-3} rad)
Exact	3.45454	3.36316
Second-order, $N = 1$	3.45454	3.36328
First-order, $N = 50$	3.45308	3.36546
First-order, $N = 1$	3.45055	3.36617

Examples

Let us consider, as a numerical example, a satellite orbit similar to the nominal orbit of the first "Telstar" satellite. In Table 1 we list the perturbations in Ω (nodal angle) and ω (argument of perigee), the elements that exhibit secular changes. The changes are those which occur as the angular variable θ (the argument of the latitude in the perturbed orbit) advances by 2π.

We see that by updating the first-order expressions 50 times we have removed 63% of the discrepancy between the first-order and exact (or second-order) results for $\Delta\Omega$, and 25% of the discrepancy for $\Delta\omega$. (All that has been shown mathematically is that the results for $N \to \infty$ approach the exact answer; not how good the results are for any given N.)

Updating more frequently should continue to improve the results until the practical limit mentioned in the previous footnote is reached. Of course, this requires more and more computation time. One must balance this against the time required to develop a higher-order perturbation scheme. (Reference 2 required more than 12 man-months of analysis and programing. Thus it may be useful to remember that the alternative discussed here is available.)

In Ref. 5, several second-order perturbation schemes are evaluated numerically. (These are Brouwer's von Zeipel approach, Izsak's Vinti method, and methods developed by Petty and Breakwell, by Hall, Gawlowicz, and Vahradian, and by the Aeronutronic staff.) The perturbation scheme VARPAR, described in Ref. 2, is also basically§ a second-

† Obviously what follows is not restricted to applications in celestial mechanics, although the examples considered will be taken from that field.

‡ In practice, a limit is reached when the increments in (13) are smaller than the number of digits carried, even when the increments are accumulated as "running sums." In theory, this limit can always be extended by multiple precision arithmetic.

§ The change in semimajor axis is obtained through terms of third-order by means of the energy integral.

order scheme. However, in VARPAR the second-order expressions are updated every period when predicting several periods ahead. Comparing Table 3 of Ref. 2 with Figs. 33 and 36 of Ref. 5, indicates that the total position error with VARPAR is at least 10 times less than the in-track error with the best of the schemes described in Ref. 5. Of course, the accuracy of any of these schemes could be improved by the updating procedure that we advocate here.

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Technical Comments

Comments on "Some Energy and Momentum Considerations in the Perforation of Plates"

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The equations governing the macroscopic behavior of a plate-projectile impact situation, commonly called plugging, were developed by Giere.[1] The two-body, dead or plastic impact equations were shown to result from an analysis of the perforation of thin plates by projectiles impacting at ordnance velocities if the kinetic energy and momentum transferred to the target are considered to be negligible. This analysis assumes that the projectile and plug leave the plate with a common velocity. The residual velocity of the projectile, compared with the initial velocity, was given as

$$v_r/v_s = (1 + \alpha)^{-1} \tag{1}$$

where α is the ratio of the mass of the plug to the mass of the projectile. Plugging has often been advocated as a model for systems wherein the materials are ductile, the projectile is blunt, the target thickness is small compared with the length of the projectile, and the projectile is initially traveling at ordnance velocities greater than the minimum perforation velocity. Although some of the restrictions on this theory were pointed out by Giere, the limits that they impose have not been demonstrated.

There are data available which can be compared with the theory of Eq. (1) if it is assumed that the projectile and plug have the same cross-sectional area.[2–5] The data used in Fig. 1 for each point were the average of at least four tests with impact velocities between 50% greater than the minimum perforation velocity and 1 km/sec, except for that of Ref. 3. Spells' data were all taken at 1.469 km/sec, and it demonstrates the trend of closer agreement with the theory at higher velocities which the other data showed.

It is apparent that these data are incompatible with the assumption of negligible energy loss from the projectile-plug system as used in the development of Eq. (1). Giere's analysis suggests that a possible explanation for this disagreement

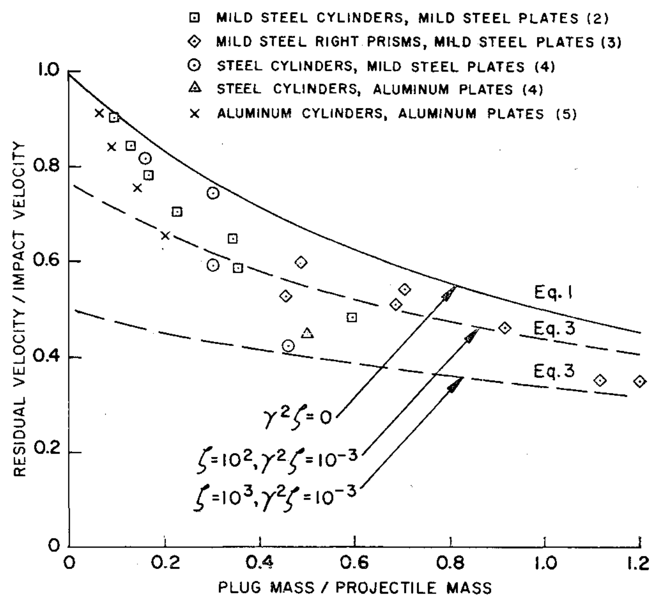

Fig. 1 Projectile velocity ratio for plate perforations.

is the nonkinetic energy going into the target E_0. However, if the plate momentum and energy are equal to zero, then $E_0 = 0$ if the energy and momentum of the system are to be conserved. If the momentum acquired by the target is not excluded from the system, the nonkinetic energy W can be expressed as

$$W = E + E_0 = \frac{mv_s^2}{2}\left[1 - \frac{1 + \alpha + \gamma^2\zeta}{(1 + \alpha + \gamma\zeta)^2}\right] \tag{2}$$

where, in addition to Giere's notation, $\gamma = V/v_r$ and $\zeta = (M - m_t)/m$. The inclusion of the plate in the system results in the following equation for the velocity ratio of the projectile:

$$v_r/v_s = (1 + \alpha + \gamma\zeta)^{-1} \tag{3}$$

Although it is realistic to assume that, in comparison with the projectile, the target acquires negligible energy, because the relative mass of the target is so great, it does not necessarily follow that its momentum is negligible. The effect of the target momentum is shown in Fig. 1 by the dashed lines that

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